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On a class of weighted anisotropic Sobolev inequalities

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1 Introduction and main results

In this article, motivated by the work of Caffarelli and Cordoba [CC] in phase transitions analysis, we prove new weighted *anisotropic* Sobolev type inequalities, that is Sobolev type inequalities where different derivatives have different weight functions.

Phase transitions or interfaces appear in physical problems when two different states coexist and there is a balance between two opposite tendencies: a diffusive effect that tends to mix the materials and a mechanism that drives them into their pure state, which is typically given by a nonnegative potential $F(x,u)$, denoting the energy density of the configuration u. For example it is known that minimizers of the functionals

$$J_\epsilon(u) := \int_\Omega \{\epsilon^2|\nabla u|^2 + F(x,u)\}dx,$$

for $0<\epsilon<1$, $F(x,u)=(1-u^2)_+^\delta$, and $\Omega\subset\mathbb{R}^N$ open and bounded, develop free boundaries if $0<\delta<2$, while generate exponential convergence to the states ±1 if $\delta=2$, that is in the case connected to the Ginzburg-Landau equation, see [CC].

The main results of [CC] are concerned with the study of regularity properties of interfaces. Their results are closely related to a conjecture of De Giorgi according to which bounded solutions of the Ginzburg-Landau scalar equation on the whole space $\mathbb{R}^N$ that are monotone in one direction, are one dimensional (see [DG]); in particular they concern the question of De Giorgi under the additional assumption that the level sets are the graphs of an equi-Lipschitz family of functions (see [MM] for the case

$N = 2$, see also [BBG] for the general case). In establishing these results a central role is played by various anisotropic Sobolev type inequalities, see Propositions 4–5 in [CC].

Moreover, the weighted anisotropic Sobolev inequalities we are dealing with, are also intimately connected to Sobolev inequalities for Grushin type operators. Unweighted local version of this type of inequalities have been studied in [FL1], [FL2], as well as in [FGW] where Muckenhoupt weights were considered.

As a further motivation to the present study, we mention that Sobolev inequalities, are used in the proof of Liouville type theorems for the corresponding linear elliptic operators in divergence form.

For other type of anisotropic Sobolev type inequalities we refer to [Ba], [Be], [Mo].

To state our results let us first introduce some notation. We define the infinite cylinder $\mathcal{H}_1$ as well as the finite cylinder $\mathcal{C}_1$ by

$$\begin{aligned}\mathcal{H}_1 &:= \{(x', \lambda) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x'| < 1\},\\ \mathcal{C}_1 &:= \{(x', \lambda) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x'| < 1,\ |\lambda| < 1\}.\end{aligned}$$

We will prove weighted Sobolev inequalities on the finite cylinder $\mathcal{C}_1$, the weight being a positive power of the distance function to the top or the bottom of the cylinder $\{\lambda = \pm 1\}$.

Our first result is the following

Theorem 1.1 *Let $N \geq 2$, $\alpha > -1$ and $\sigma \in (-2\alpha, 2)$. Then, for any Q with*

$$2 \leq Q \leq Q_{cr}(N, \alpha, \sigma) := \frac{2\left(N + \frac{2\alpha+\sigma}{2-\sigma}\right)}{N + \frac{2\alpha+\sigma}{2-\sigma} - 2}, \tag{1.1}$$

there exists a positive constant $C = C(Q, N, \alpha, \sigma)$, such that for any function $f \in C_0^\infty(\mathcal{H}_1)$ there holds

$$\left(\int_{\mathcal{C}_1} (1 - |\lambda|)^\alpha |f(x', \lambda)|^Q dx' d\lambda\right)^{\frac{2}{Q}} \leq C \int_{\mathcal{C}_1} (1 - |\lambda|)^\alpha \left(|\nabla_{x'} f|^2 + (1 - |\lambda|)^\sigma |\partial_\lambda f|^2\right) dx' d\lambda. \tag{1.2}$$

In the limit case where $\sigma = 2$, estimate (1.2) holds for $Q = 2$ and any $f \in C_0^\infty(\mathcal{H}_1)$ but fails for $Q > 2$ and $f \in C_0^\infty(\mathcal{C}_1)$.

In the case $\sigma \geq 2$ we can still have similar inequalities when $\alpha < -1$. More precisely when $\sigma = 2$ we have

Theorem 1.2 *Let $N \geq 2$ and $\alpha < -1$. For any Q with $2 \leq Q \leq \frac{2N}{N-2}$, in case $N \geq 3$, or $Q \geq 2$ in case $N = 2$, there exists a positive constant $C = C(N, \alpha, Q)$, such that for any function $f \in C_0^\infty(\mathcal{C}_1)$ there holds*

$$\left(\int_{\mathcal{C}_1} (1 - |\lambda|)^\alpha |f(x', \lambda)|^Q dx' d\lambda\right)^{\frac{2}{Q}} \leq C \int_{\mathcal{C}_1} (1 - |\lambda|)^\alpha \left(|\nabla_{x'} f|^2 + (1 - |\lambda|)^2 |\partial_\lambda f|^2\right) dx' d\lambda. \tag{1.3}$$

When $\sigma > 2$ we obtain the same inequality but this time for exponents Q that satisfy $Q \geq Q_{cr}$ as defined in (1.1). Thus, we have

Theorem 1.3 *Let $N \geq 2$, $\alpha < -1$ and $\sigma \in (2, -2\alpha)$. Then, for any Q with $Q_{cr} \leq Q$ if $N = 2$ or $Q_{cr} \leq Q \leq \frac{2N}{N-2}$ if $N \geq 3$, there exists a positive constant $C = C(N, Q, \alpha, \sigma)$, such that for any function $f \in C_0^\infty(\mathcal{C}_1)$ there holds*

$$\left(\int_{\mathcal{C}_1} (1 - |\lambda|)^\alpha |f(x', \lambda)|^Q dx' d\lambda\right)^{\frac{2}{Q}} \leq C \int_{\mathcal{C}_1} (1 - |\lambda|)^\alpha \left(|\nabla_{x'} f|^2 + (1 - |\lambda|)^\sigma |\partial_\lambda f|^2\right) dx' d\lambda. \tag{1.4}$$

When $\alpha > -1$ then $(1-|\lambda|)^\alpha$ is an $L^1(-1,1)$ function and using Holder's inequality one can obtain the inequality for any Q with $2 \le Q \le Q_{cr}$ once it is true for Q_{cr}. However this is not the case when $\alpha < -1$.

We note that for $Q = 2$ inequality (1.4) is still valid as one can see using Poincaré inequality in the x'–variables. The validity or not of (1.4) for $2 < Q < Q_{cr}$ remains an open question.

Finally, as $\sigma > 2$ approaches 2, Q_{cr} approaches 2 and therefore the Q–interval of validity of (1.4) approaches the interval $[2, \frac{2N}{N-2}]$ in complete agreement with the result of Theorem 1.2.

A central role in the proof of the previous results, is played by various weighted isotropic Sobolev inequalities in the upper half space $\mathbb{R}^N_+ := \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > 0\}$, which are of independent interest. We present such a result:

Theorem 1.4 *Let either*

$$N = 2, \qquad 2 \le Q, \qquad \text{and} \qquad B = A - \frac{2}{Q}\ , \tag{1.5}$$

or else,

$$N \ge 3, \qquad 2 \le Q \le \frac{2N}{N-2}, \qquad \text{and} \qquad B = A - 1 + \frac{Q-2}{2Q} N. \tag{1.6}$$

If $BQ + 2A \neq 0$, or if $A = B = 0$ then
(i)There exists a positive constant $C = C(A, Q, N)$, such that for any function $f \in C_0^\infty(\mathbb{R}^N_+)$ there holds

$$\left(\int_{\mathbb{R}^N_+} x_N^{BQ} |f(x', x_N)|^Q dx' dx_N \right)^{\frac{2}{Q}} \le C \int_{\mathbb{R}^N_+} x_N^{2A} \left(|\nabla_{x'} f|^2 + |\partial_{x_N} f|^2 \right) \, dx' dx_N\ . \tag{1.7}$$

(ii) If moreover $BQ + 2A > 0$, or if $A = B = 0$ inequality (1.7) still holds even if $f \in C_0^\infty(\mathbb{R}^N)$.

The exponent $Q = Q(A, B, N)$ given by conditions (1.6) and (1.5) is the best possible, as one can easily see arguing by scaling $x' = Ry'$, $x_N = Ry_N$. In case $N \ge 3$, part (i) of the Theorem 1.4 is due to Maz'ya, see [M], section 2.1.6. Here we will provide a simpler proof along the lines of [FMaT1], [FMaT2], [FMT]. A particular case of (1.7) has been obtained in [C] under an additional assumption on f, by different methods.

We next present a direct consequence of Theorem 1.1.

Corollary 1.5 *For $N \ge 2$, $m > -1$ and $\epsilon \in (0, \frac{1}{2})$ we set*

$$C_{1,\epsilon} := \{(x', \lambda) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x'| < 1, |\lambda| < 1 - \epsilon^{1+m}\}\ .$$

Let $\alpha > -1$ and $\beta > 0$ satisfy

$$-2\alpha(1+m) < \beta m < 2(1+m),$$

and

$$2 \le P \le P_{cr}(N, m, \alpha, \beta) := \frac{2\left(N + \frac{2\alpha(1+m)+\beta m}{2(1+m)-\beta m}\right)}{N + \frac{2\alpha(1+m)+\beta m}{2(1+m)-\beta m} - 2}.$$

Then, there exists a positive constant $C = C(N, P, m, \alpha, \beta)$ independent of ϵ, such that for any function $f \in C_0^\infty(C_{1,\epsilon})$ there holds

$$\left(\int_{C_{1,\epsilon}} (1-|\lambda|)^\alpha |f(x', \lambda)|^P dx' d\lambda \right)^{\frac{2}{P}} \le C \int_{C_{1,\epsilon}} (1-|\lambda|)^\alpha \left(|\nabla_{x'} f|^2 + \frac{(1-|\lambda|)^\beta}{\epsilon^\beta} |\partial_\lambda f|^2 \right) dx' d\lambda.$$

The above corollary is in the same spirit as the results in [CC]. Indeed, when $\alpha = 1$ and $\beta = 2$, Corollary 1.5 entails the weighted Sobolev inequality of Proposition 5 of [CC] providing a precise range for the Sobolev exponent. Analogous results can be easily obtained in case $\alpha < -1$, by using Theorems 1.2 and 1.3.

We next consider the more general case of weighted anisotropic inequalities where the distance is taken from a higher codimension boundary. More precisely, for $x \in \mathbb{R}^N$ we write $x = (x', \lambda)$, with $x' \in \mathbb{R}^{N-k}$ and $\lambda \in \mathbb{R}^k$, with $1 < k < N$. Let $\Omega \subset \mathbb{R}^k$ be a smooth bounded domain and $B_1 = \{x' : |x'| < 1\}$ be the unit ball in $\mathbb{R}^{N-k}$. We also set $d = d(\lambda) = \mathrm{dist}(\lambda, \partial\Omega)$. In this case our main result reads

Theorem 1.6 *Let $N \geq 3$, $1 < k < N$, $\alpha > -1$ and $\sigma \in (-2\alpha, 2)$ with $2\alpha + \sigma k \geq 0$. Then, for any Q,*

$$2 \leq Q \leq Q^k_{cr} := \frac{2(N + \frac{2\alpha+\sigma k}{2-\sigma})}{N + \frac{2\alpha+\sigma k}{2-\sigma} - 2},$$

there exists a positive constant $C = C(Q, N, \alpha, \sigma, k)$, such that for any function $f \in C_0^\infty(B_1 \times \Omega)$ there holds

$$\left(\int_{B_1\times\Omega} d^\alpha |f|^Q dx\right)^{\frac{2}{Q}} \leq C \int_{B_1\times\Omega} d^\alpha \left(|\nabla_{x'} f|^2 + d^\sigma |\nabla_\lambda f|^2\right) dx. \tag{1.8}$$

The limit case $k = N$, corresponds to the following isotropic weighted inequality

$$\left(\int_\Omega d^\alpha |f|^Q d\lambda\right)^{\frac{2}{Q}} \leq C \int_\Omega d^{\alpha+\sigma} |\nabla_\lambda f|^2 d\lambda,$$

which is true when $\alpha + \sigma < 1$ but not when $\alpha + \sigma \geq 1$; see Remark after the proof of Theorem 1.6 for details.

To prove the above Theorem an important role is played by the following weighted anisotropic Sobolev inequality in the upper half space $\mathbb{R}^N_+$. To state the result we first introduce some notation. For $x \in \mathbb{R}^N_+$, $1 < k < N$, we write $x = (x', \lambda) = (x', x_N, y)$, with $x' \in \mathbb{R}^{N-k}$, $x_N \in [0, \infty)$, and $y \in \mathbb{R}^{k-1}$. We also write dx for $dx' d\lambda = dx' dx_N dy$.

Theorem 1.7 *Let $\gamma \in \mathbb{R}$, and either*

$$N = 2, \qquad Q \geq 2, \qquad \text{and} \qquad B = A - 1 + \frac{Q-2}{2Q}(2 + \gamma(k-1)), \tag{1.9}$$

or else,

$$N \geq 3, \qquad 2 \leq Q \leq \frac{2N}{N-2}, \qquad \text{and} \qquad B = A - 1 + \frac{Q-2}{2Q}(N + \gamma(k-1)). \tag{1.10}$$

If $BQ + 2A \neq 0$ then
(i)There exists a positive constant $C = C(A, Q, N, k, \gamma)$, such that for any function $f \in C_0^\infty(\mathbb{R}^N_+)$ there holds

$$\left(\int_{\mathbb{R}^N_+} x_N^{BQ} |f(x)|^Q dx\right)^{\frac{2}{Q}} \leq C \int_{\mathbb{R}^N_+} x_N^{2A} \left(|\nabla_{x', x_N} f|^2 + x_N^{2\gamma} |\nabla_y f|^2\right) \, dx \; . \tag{1.11}$$

(ii) If moreover $BQ + 2A > 0$, inequality (1.11) still holds even if $f \in C_0^\infty(\mathbb{R}^N)$.

We note that the exponent $Q = Q(A, B, N, \gamma, k)$ given by (1.10) is the best possible as one can easily check using the natural scaling $x' = Rz'$, $x_N = Rz_N$ and $y = R^{\gamma+1}w$.

Inequality (1.11) is a weighted Sobolev inequality for Grushin type operators $\mathcal{L}_\gamma := \Delta_{x',x_N} + x_N^{2\gamma}\Delta_y$ having associated gradient $\nabla_\gamma := (\nabla_{x'}, \partial_{x_N}, x_N^\gamma \nabla_y)$, so that

$$|\nabla_\gamma g|^2 = |\nabla_{x',x_N} g|^2 + x_N^{2\gamma}|\nabla_y g|^2 \ .$$

When $\gamma \in \mathbb{N}$ then $\mathcal{L}_\gamma := \frac{\partial^2}{\partial x_1^2} + x_1^{2\gamma}\frac{\partial^2}{\partial x_2^2}$ belongs to the class of differential operators considered by [B]; in particular, it is hypoelliptic and satisfies a Harnack inequality since the Lie algebra generated by the vector fields $\frac{\partial}{\partial x_1}$ and $x_1^\gamma \frac{\partial}{\partial x_2}$ has rank two at any point of the plane. On the other hand when $\gamma > 0$ and $-1 < 2A < 1$, the weight is a Muckehoupt weight and the local version of inequality (1.11) was considered in [FGW]. Our method has the advantage of allowing a bigger range of values for the parameter A, in particular allowing weights outside the Muckehoupt classes.

We finally note that weighted Sobolev type inequalities of the kind we present in this work, play an important role in establishing Harnack inequalities and heat kernel estimates in [FMT] in the isotropic case, whereas in the non isotropic case, weighted Sobolev inequalities are crucial in establishing Liouville type Theorems, see [CM].

This paper is organized as follows. In Sections 2, 3 and 4 we consider the case of codimension $k = 1$ case. In particular, in Section 2 we study the case $\sigma < 2$, in Section 3 the critical case $\sigma = 2$, whereas in Section 4 the supercritical case $\sigma > 2$. Finally the last Section 5 is devoted to the study of the higher codimension case and in particular we give the proofs of Theorems 1.6 and 1.7.

2 Codimension 1 degeneracy; the case $\sigma < 2$.

In this Section we will give the proofs of Theorems 1.1, 1.4 and Corollary 1.5.

We first give the proof Theorem 1.4.

Proof of Theorem 1.4: Let us first give the proof of part (ii). For any $u \in C_0^\infty(\mathbb{R}^N)$ it is well known that

$$S_N ||u||_{L^{\frac{N}{N-1}}} \leq ||\nabla u||_{L^1} \ , \tag{2.12}$$

where $S_N := N\pi^{\frac{1}{2}}\left[\Gamma(1+\frac{N}{2})\right]^{-\frac{1}{2}}$ (see, e.g., p. 189 in [M]). We apply (2.12) to the function $u := x_N^a v$, for $v \in C_0^\infty(\mathbb{R}^N)$ and $a > 0$. Thus, we have

$$S_N ||x_N^a v||_{L^{\frac{N}{N-1}}} \leq \int_{\mathbb{R}^N_+} \left(|\nabla v| x_N^a + a x_N^{a-1}|v|\right) dx' dx_N \ .$$

To estimate the last term of the right hand side, we integrate by parts,

$$a\int_{\mathbb{R}^N_+} x_N^{a-1}|v| dx' dx_N = \int_{\mathbb{R}^N_+} \nabla x_N^a |v| dx' dx_N = -\int_{\mathbb{R}^N_+} x_N^a \nabla |v| dx' dx_N. \tag{2.13}$$

From this we get

$$a\int_{\mathbb{R}^N_+} x_N^{a-1}|v| dx' dx_N \leq \int_{\mathbb{R}^N_+} |\nabla v| x_N^a dx' dx_N. \tag{2.14}$$

Consequently,

$$||x_N^a v||_{L^{\frac{N}{N-1}}} \leq 2S_N^{-1}\int_{\mathbb{R}^N_+} |\nabla v| x_N^a dx' dx_N. \tag{2.15}$$

For any $1 \le p \le \frac{N}{N-1}$ and any two functions w and v, the following interpolation inequality can be easily seen to be true:

$$||w^b v||_{L^p} \le C_1 ||w^a v||_{L^{\frac{N}{N-1}}} + C_2 ||w^{a-1} v||_{L^1}\ , \quad \text{for} \quad b = a - 1 + \frac{p-1}{p} N, \tag{2.16}$$

with two positive constants C_1, C_2 independent of w and v.

From (2.15) and (2.16) for $w := x_N$ we obtain the following

$$\left(\int_{\mathbb{R}^N_+} x_N^{bp} |v|^p dx' dx_N \right)^{\frac{1}{p}} \le C_1 \int_{\mathbb{R}^N_+} |\nabla v| x_N^a dx' dx_N + C_2 \int_{\mathbb{R}^N_+} x_N^{a-1} |v| dx' dx_N. \tag{2.17}$$

Using now (2.14) we arrive at the following $L^p - L^1$ weighted estimate

$$\left(\int_{\mathbb{R}^N_+} x_N^{bp} |v|^p dx' dx_N \right)^{\frac{1}{p}} \le C_1 \int_{\mathbb{R}^N_+} |\nabla v| x_N^a dx' dx_N. \tag{2.18}$$

To pass to the corresponding $L^Q - L^2$ estimate we apply (2.18) to $v := |f|^s$, $s > 0$, to obtain

$$\left(\int_{\mathbb{R}^N_+} x_N^{bp} |f|^{ps} dx' dx_N \right)^{\frac{1}{p}} \le C \int_{\mathbb{R}^N_+} f^{s-1} |\nabla f| x_N^a dx' dx_N =$$

$$= C \int_{\mathbb{R}^N_+} x_N^{\frac{bp}{2}} |f|^{s-1} |\nabla f| x_N^{a - \frac{bp}{2}} dx' dx_N \le$$

$$\le C \left(\int_{\mathbb{R}^N_+} x_N^{bp} |f|^{2s-2} dx' dx_N \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N_+} |\nabla f|^2 x_N^{2a-bp} dx' dx_N \right)^{\frac{1}{2}}.$$

Choosing $s = \frac{2}{2-p}$ so that $2s - 2 = ps$ we get

$$\left(\int_{\mathbb{R}^N_+} x_N^{bp} |f|^{ps} dx' dx_N \right)^{\frac{2}{p}-1} \le C \int_{\mathbb{R}^N_+} x_N^{2a-bp} |\nabla f|^2 dx' dx_N. \tag{2.19}$$

To arrive at (1.7) we take $BQ = bp$, $Q = ps$ and $2a - bp = 2A$. For this choice of the parameters we arrive at

$$\left(\int_{\mathbb{R}^N_+} x_N^{BQ} |f|^Q dx' dx_N \right)^{\frac{2}{Q}} \le C \int_{\mathbb{R}^N_+} x_N^{2A} |\nabla f|^2 dx' dx_N$$

with $2 \le Q \le \frac{2N}{N-2}$ and $B = A - 1 + \frac{Q-2}{2Q} N$, in case $N \ge 3$, or $Q \ge 2$ and $B = A - \frac{2}{Q}$ in case $N = 2$.

Since $2a = 2A + BQ$, the condition $a > 0$ is equivalent to $BQ + 2A > 0$. This completes the proof of part (ii) of Theorem 1.4.

Concerning part (i), we note that for $v \in C_0^\infty(\mathbb{R}^N_+)$, and $a \in \mathbb{R}$, it follows from (2.13) that

$$|a| \int_{\mathbb{R}^N_+} x_N^{a-1} |v| dx' dx_N \le \int_{\mathbb{R}^N_+} |\nabla v| x_N^a dx' dx_N. \tag{2.20}$$

Consequently, estimate (2.15) remains true for any $a \in \mathbb{R}$. Estimate (2.17) is still true, and using (2.20) we arrive at (2.18). The use of (2.20) however imposes the condition that $a \neq 0$. The rest of the argument remains the same. The condition $a \neq 0$ is equivalent to $BQ + 2A \neq 0$.

We finally note that, when $A = B = 0$ then (1.7) is the standard Sobolev inequality.

■

As a consequence of the Theorem 1.4 we have the following inequality in a strip:

Proposition 2.1 *Let* $\mathcal{H}_1 = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x'| < 1\}$,

$$N = 2, \qquad 2 \le Q, \qquad \text{and} \qquad B = A - \frac{2}{Q},$$

or

$$N \ge 3, \qquad 2 \le Q \le \frac{2N}{N-2}, \qquad \text{and} \qquad B = A - 1 + \frac{Q-2}{2Q} N.$$

If $BQ + 2A \neq 0$, *or if* $A = B = 0$ *then,*
(i) There exists a positive constant $C = C(A, Q, N)$, *such that for any function* $f \in C_0^\infty(\mathcal{H}_1 \cap \mathbb{R}^N_+)$ *there holds*

$$\left(\int_{\mathcal{H}_1 \cap \{0 < x_N < 1\}} x_N^{BQ} |f(x', x_N)|^Q dx' dx_N \right)^{\frac{2}{Q}} \le C \int_{\mathcal{H}_1 \cap \{0 < x_N < 1\}} x_N^{2A} \left(|\nabla_{x'} f|^2 + |\partial_{x_N} f|^2 \right) \, dx' dx_N \ , \quad (2.21)$$

(ii) If moreover $BQ + 2A > 0$, *inequality (2.21) still holds even if* $f \in C_0^\infty(\mathcal{H}_1)$

In the case where $2A = BQ \in (0, \infty)$ and under the more restrictive assumption that $f \in C_0^\infty(\mathcal{H}_1 \cap \{0 < x_N < 1\})$, the result of part (ii) has been established in [C] by different methods (see also [CF]).

Proof of Proposition 2.1: We prove part (ii), the other case being quite similar. In order to do this due to Theorem 1.4 part (ii) it is enough to remove zero boundary conditions on the hyperplane $x_N = 1$. Let $f \in C_0^\infty(\mathcal{H}_1)$ and we denote by $\xi(x_N)$ a C^1 function such that $\xi(x_N) = 1$ if $x_N \le \frac{1}{2}$ and $\xi(x_N) = 0$ if $x_N \ge 1$. We then have

$$\begin{aligned} LHS \ &:= \ C \left(\int_{\{0 < x_N < 1\}} x_N^{BQ} |f|^Q dx' dx_N \right)^{\frac{2}{Q}} \\ &\le \ \left(\int_{\{0 < x_N < 1\}} x_N^{BQ} |f\xi|^Q dx' dx_N \right)^{\frac{2}{Q}} + \left(\int_{\{\frac{1}{2} < x_N < 1\}} x_N^{BQ} |f(1-\xi)|^Q dx' dx_N \right)^{\frac{2}{Q}} \\ &=: \ I_1 + I_2. \end{aligned} \quad (2.22)$$

Applying Theorem 1.4 part (ii) to the function $f\xi$, we obtain

$$\begin{aligned} I_1 \ &\le \ C \int_{\{0 < x_N < 1\}} x_N^{2A} \left(|\nabla_{x'} (f\xi)|^2 + |\partial_{x_N} (f\xi)|^2 \right) \, dx' dx_N \\ &\le \ C \int_{\{0 < x_N < 1\}} x_N^{2A} \left(|\nabla_{x'} f|^2 + |\partial_{x_N} f|^2 + f^2 \right) \, dx' dx_N. \end{aligned} \quad (2.23)$$

Concerning I_2 we note that the weights x_N^{BQ} and x_N^{2A} are uniformly bounded both from above and below for $x_N \in [\frac{1}{2}, 1]$, and therefore, applying the standard Sobolev inequality to the function $f(1-\xi)$ which is zero for $|x'| = 1$ as well as for $x_N = \frac{1}{2}$ we get

$$I_2 \le C \int_{\{\frac{1}{2} < x_N < 1\}} x_N^{2A} \left(|\nabla_{x'} f|^2 + |\partial_{x_N} f|^2 + f^2 \right) \, dx' dx_N.$$

Combining this with (2.22) and (2.23) we get

$$LHS \le C \int_{\{0<x_N<1\}} x_N^{2A} \left(|\nabla_{x'} f|^2 + |\partial_{x_N} f|^2 + f^2 \right) \, dx' dx_N. \tag{2.24}$$

To continue, let $B_1' := \{x' \in \mathbb{R}^{N-1} : |x'| < 1\}$. For any fixed $x_N \in [0,1]$, we have by the Poincaré inequality

$$\int_{B_1'} f^2(x', x_N) dx' \le C \int_{B_1'} |\nabla_{x'} f|^2 dx',$$

whence

$$\int_0^1 \int_{B_1'} f^2(x', x_N) dx' x_N^{2A} dx_N \le C \int_0^1 \int_{B_1'} |\nabla_{x'} f|^2 dx' x_N^{2A} dx_N \ .$$

From this and (2.24) the result follows. ∎

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1: It is enough to prove (1.2) in the upper half cylinder; that is, if $f \in C_0^\infty(\mathcal{H}_1)$ then we will show that

$$\left(\int_{\{0<\lambda<1\}} (1-\lambda)^\alpha |f(x',\lambda)|^Q dx' d\lambda \right)^{\frac{2}{Q}} \le C \int_{\{0<\lambda<1\}} (1-\lambda)^\alpha \left(|\nabla_{x'} f|^2 + (1-\lambda)^\sigma |\partial_\lambda f|^2 \right) dx' d\lambda \ . \tag{2.25}$$

We first consider the case $\sigma < 2$. We change variables by $x' = x'$, $s = (1-\lambda)^{\frac{2-\sigma}{2}}$ thus setting $\varphi(x', s) := f(x', 1 - s^{\frac{2}{2-\sigma}})$, it follows that inequality (2.25) is equivalent to

$$\left(\int_{\{0<s<1\}} s^{\frac{\sigma+2\alpha}{2-\sigma}} |\varphi(x',s)|^Q dx' ds \right)^{\frac{2}{Q}} \le C \int_{\{0<s<1\}} s^{\frac{\sigma+2\alpha}{2-\sigma}} \left(|\nabla_{x'} \varphi|^2 + |\partial_s \varphi|^2 \right) \, dx' ds \ , \tag{2.26}$$

in fact we easily compute that $ds = \frac{\sigma-2}{2}(1-\lambda)^{-\frac{\sigma}{2}} d\lambda$, $\partial_\lambda = \frac{ds}{d\lambda}\partial_s = \frac{\sigma-2}{2}(1-\lambda)^{-\frac{\sigma}{2}}\partial_s$ and

$$|\nabla_{x'} f|^2 + (1-\lambda)^\sigma |\partial_\lambda f|^2 = |\nabla_{x'} \varphi|^2 + \left(\frac{\sigma-2}{2} \right)^2 |\partial_s \varphi|^2 \ .$$

When $\sigma \in (-2\alpha, 2)$ we now use Proposition 2.1, part (ii). Suppose first that $N \ge 3$. For $A = \frac{\sigma+2\alpha}{2(2-\sigma)}$ and $B = \frac{\sigma+2\alpha}{2(2-\sigma)} - 1 + \frac{Q-2}{2Q}N$, with $2 \le Q \le \frac{2N}{N-2}$ we have that the right hand side of (2.26) dominates

$$\left(\int_{\{0<s<1\}} s^{BQ} |\varphi(x',s)|^Q dx' ds \right)^{\frac{2}{Q}} .$$

To deduce (2.26) we need $\frac{\sigma+2\alpha}{2-\sigma} \ge BQ = \left(\frac{\sigma+2\alpha}{2(2-\sigma)} - 1 + \frac{Q-2}{2Q}N \right) Q$, which is equivalent to

$$Q \le 2 \frac{N + \frac{2\alpha+\sigma}{2-\sigma}}{N + \frac{2\alpha+\sigma}{2-\sigma} - 2}. \tag{2.27}$$

On the other hand, the restriction $2A + BQ > 0$ is easily seen to be equivalent to

$$Q > 2 \frac{N - \frac{2\alpha+\sigma}{2-\sigma}}{N + \frac{2\alpha+\sigma}{2-\sigma} - 2} =: \bar{Q}. \tag{2.28}$$

We note that Q_{cr} as given by (1.1) satisfies both (2.27) and (2.28) and therefore (1.2) has been proved for $Q = Q_{cr}$. The full range of Q follows by using Holder's inequality in the left hand side of (1.2).

The case $N = 2$ is treated quite similarly. Thus (1.2) has been proved for any $f \in C_0^\infty(\mathcal{H}_1)$.

In the special case $\sigma = 2$ and $Q = 2$ we note that (1.2) is still valid. To see this we change variables by $x' = x'$ and $t = (1-\lambda)^{\alpha+1}$ thus setting $g(x', t) := f(x', 1 - t^{\frac{1}{\alpha+1}})$. It follows that inequality (2.25) is equivalent to

$$\left(\int_{t\in(0,1)} |g(x',t)|^Q dx'dt \right)^{\frac{2}{Q}} \leq C \int_{t\in(0,1)} \left(|\nabla_{x'} g|^2 + t^2 |\partial_t g|^2 \right) \, dx'dt \ , \tag{2.29}$$

in fact we easily compute that $dt = -(\alpha+1)(1-\lambda)^\alpha d\lambda$, $\partial_\lambda = \frac{dt}{d\lambda}\partial_t = -(\alpha+1)(1-\lambda)^\alpha \partial_t$ and

$$|\nabla_{x'} f|^2 + (1-\lambda)^\sigma |\partial_\lambda f|^2 = |\nabla_{x'} g|^2 + (\alpha+1)^2 \, t^2 |\partial_t g|^2 \ .$$

Inequality (2.29) with $Q = 2$ holds true, as one can easily see using Poincaré inequality for the slices $t =$ constant.

It remains to show that (1.2) fails in the case $\sigma = 2$, $\alpha > -1$ and $Q > 2$ even thought we take $f \in C_0^\infty(\mathcal{C}_1)$. To this end, let us make use of the following different change of variables $x' = x'$ and $\lambda = \tanh x_N$. Then $\lambda \in (-1, 1)$ goes to $x_N \in (-\infty, \infty)$ and $(1 - |\lambda|) \sim (1 - \lambda^2) = (\cosh x_N)^{-2} \sim e^{-2|x_N|}$ and $d\lambda \sim (\cosh x_N)^{-2} dx_N \sim e^{-2|x_N|} dx_N$. We define $g(x', x_N) := f(x', \tan h x_N)$. Then it follows from that for any function $g \in C_0^\infty(\mathcal{H}_1)$ the following inequality should be true if (1.2) holds true:

$$\left(\int_{\mathcal{H}_1} e^{-2(\alpha+1)|x_N|} |g(x', x_N)|^Q dx'dx_N \right)^{\frac{2}{Q}} \leq C \int_{\mathcal{H}_1} e^{-2(\alpha+1)|x_N|} \left(|\nabla_{x'} g|^2 + |\partial_{x_N} g|^2 \right) dx'dx_N \ . \tag{2.30}$$

For $g \in C_0^\infty(\mathcal{H}_1 \cap \{x_N > 0\})$ we set $g_\tau(x', x_N) := g(x', x_N - \tau)$, $\tau > 0$. Clearly, $g_\tau \in C_0^\infty(\mathcal{H}_1 \cap \{x_N > 0\})$ and applying (2.30) to the family g_τ we get

$$\left(\int_{\mathcal{H}_1} e^{-2(\alpha+1)x_N} |g(x', x_N)|^Q dx'dx_N \right)^{\frac{2}{Q}} \leq C e^{-2\tau(\alpha+1)\left(\frac{Q-2}{Q}\right)} \int_{\mathcal{H}_1} e^{-2(\alpha+1)x_N} \left(|\nabla_{x'} g|^2 + |\partial_{x_N} g|^2 \right) dx'dx_N \ ,$$

for any $\tau > 0$. Taking the limit $\tau \to +\infty$ we reach a contradiction for $Q > 2$, $\alpha > -1$.

This completes the proof of Theorem 1.1.

∎

Remark. In case $\sigma = 2\alpha$ and $\alpha \in (0, 1)$, estimate (1.2) is an improvement of Proposition 4 of Caffarelli and Cordoba [CC]. Indeed, our Sobolev exponent Q_{cr} is strictly bigger than the one coming from the arguments of [CC] – which is less than $\frac{2N}{N + \frac{4\alpha}{\alpha+1} - 2}$. Moreover, we only assume that $f \in C_0^\infty(\mathcal{H}_1)$ instead of $f \in C_0^\infty(\mathcal{C}_1)$.

Remark. In case $\sigma = -\alpha$ and $\alpha > 0$ inequality (1.2) is a Sobolev inequality for a Grushin type operator corresponding to the vector fields $((1 - |\lambda|)^{\frac{\alpha}{2}} \nabla_{x'}, \partial_\lambda)$; we refer to [FL2] where local versions of similar inequalities have been considered.

Remark. We note that in the case $\sigma = -2\alpha$, estimate (2.26) corresponds to the standard Sobolev inequality in a strip, and the result follows from Proposition 2.1 part (i); thus (1.2) still holds true for any $f \in C_0^\infty(\mathcal{C}_1)$ if $\sigma = -2\alpha$.

We next show how Corollary 1.5 follows from Theorem 1.1.

Proof of Corollary 1.5: It is a consequence of Theorem 1.1. Indeed, for $(x', \lambda) \in C_{1,\epsilon}$ we have $1 - |\lambda| > \epsilon^{1+m}$, that is, $\epsilon^{-1} > (1 - |\lambda|)^{-\frac{1}{1+m}}$, and so $\frac{(1-|\lambda|)^\beta}{\epsilon^\beta} > (1 - |\lambda|)^{\frac{\beta m}{1+m}}$, $\beta > 0$. The result then follows from Theorem 1.1 by choosing $\sigma := \frac{\beta m}{1+m}$ there; in particular $P_{cr}(N, m, \alpha, \beta) = Q_{cr}(N, \alpha, \frac{\beta m}{1+m})$.

∎

3 The critical case $\sigma = 2$.

As we have seen in Theorem 1.1 inequality (1.2) fails for $\sigma = 2$, $\alpha > -1$ and $Q > 2$. To obtain Sobolev type inequalities in this case, we need to use different weights in the two sides of the inequality. More precisely we have the following

Theorem 3.1 *Let $N \geq 2$, and $\alpha > -1$. For any Q with $2 \leq Q \leq \frac{2N}{N-2}$, in case $N \geq 3$, or $Q \geq 2$ in case $N = 2$, and for any $\theta > \frac{(Q-2)(\alpha+1)}{2}$ there exists a positive constant $C = C(N, \alpha, Q, \theta)$, such that for any function $f \in C_0^\infty(\mathcal{C}_1)$ there holds*

$$\left(\int_{\mathcal{C}_1} (1-|\lambda|)^{\alpha+\theta} |f(x',\lambda)|^Q dx' d\lambda\right)^{\frac{2}{Q}} \leq C \int_{\mathcal{C}_1} (1-|\lambda|)^\alpha \left(|\nabla_{x'} f|^2 + (1-|\lambda|)^2 |\partial_\lambda f|^2\right) dx' d\lambda. \tag{3.1}$$

Proof : It is enough to prove (3.1) in the upper half cylinder; that is, if $f \in C_0^\infty(\mathcal{C}_1)$ then we will show that

$$\left(\int_{\{0<\lambda<1\}} (1-\lambda)^{\alpha+\theta} |f(x',\lambda)|^Q dx' d\lambda\right)^{\frac{2}{Q}} \leq C \int_{\{0<\lambda<1\}} (1-\lambda)^\alpha \left(|\nabla_{x'} f|^2 + (1-\lambda)^2 |\partial_\lambda f|^2\right) dx' d\lambda \ . \tag{3.2}$$

We change variables by $x' = x'$, $s = -\frac{1}{K} \ln(1-\lambda)$, for an arbitrary $K > 0$, thus setting $\varphi(x', s) = f(x', 1 - e^{-Ks})$, and arguing as in the proof of Theorem 1.1, we see that inequality (3.2) follows as soon as we prove the following inequality

$$\left(\int_{\{s>0\}} e^{-sK(\alpha+\theta+1)} |\varphi(x',s)|^Q dx' ds\right)^{\frac{2}{Q}} \leq C \int_{\{s>0\}} |\nabla\varphi|^2 e^{-sK(\alpha+1)} dx' ds \ . \tag{3.3}$$

In fact we easily compute that $d\lambda = Ke^{-Ks} ds = K(1-\lambda) ds$, $\partial_\lambda = \frac{ds}{d\lambda}\partial_s = \frac{1}{K}(1-\lambda)^{-1}\partial_s$ and

$$|\nabla_{x'} f|^2 + (1-\lambda)^2 |\partial_\lambda f|^2 = |\nabla_{x'}\varphi|^2 + \frac{1}{K^2}|\partial_s \varphi|^2 \sim |\nabla\varphi|^2 \ .$$

We note that $\varphi \in C_0^\infty(\mathcal{H}_1)$.

To continue, we will make use of Proposition 3.3 see below. For $A = \frac{K(\alpha+1)}{2}$ and $B = \frac{K(\alpha+1)}{2} + \frac{Q-2}{2Q}N = \frac{1}{Q}\left(K(\alpha+1) + \frac{(N+K(\alpha+1))(Q-2)}{2}\right)$ we have

$$\left(\int_{\mathbb{R}^N_+} e^{-sK(\alpha+1)} e^{-\theta K s} |\varphi(x',s)|^Q dx' ds\right)^{\frac{2}{Q}} \leq C \int_{\mathbb{R}^N_+} e^{-sK(\alpha+1)} |\nabla\varphi|^2 dx' ds \ , \qquad \forall \varphi \in C_0^\infty(\mathcal{H}_1), \tag{3.4}$$

where $\theta := (\frac{N}{K} + \alpha + 1)\frac{Q-2}{2}$. Note that $\theta = 0$ if $Q = 2$ as suggested by Theorem 1.1. Due to the arbitrariness of K this means that we may take any value $\theta > \frac{(\alpha+1)(Q-2)}{2}$. The restriction $2A + BQ \neq 0$ is easily seen to be equivalent to $\frac{Q+2}{2}\left(K(\alpha+1) + \frac{Q-2}{Q+2}N\right) \neq 0$, which is trivially satisfied.

The case $N = 2$ is treated quite similarly.

∎

According to Theorem 3.1 one cannot match the weights in the weighted anisotropic Sobolev inequality (3.1) when $\alpha > -1$ and $Q > 2$. However, in the case $\alpha < -1$ we can match the weights, thus proving Theorem 1.2.

Proof of Theorem 1.2 : The case $Q = 2$ is a simple consequence of Poincaré inequality. We therefore consider the case $Q > 2$. Using the same change of variables as in the proof of Theorem 3.1 the sought for inequality is equivalent to the following inequality

$$\left(\int_{\mathbb{R}^N_+} e^{-sK(\alpha+1)}|\varphi(x',s)|^Q dx'ds\right)^{\frac{2}{Q}} \le C\int_{\mathbb{R}^N_+} e^{-sK(\alpha+1)}|\nabla\varphi|^2 dx'ds\ , \qquad \forall \varphi \in C_0^\infty(\mathcal{H}_1). \tag{3.5}$$

We will use Proposition 3.5. Thus, we have

$$\left(\int_{\mathbb{R}^N_+} e^{BQs}|\varphi|^Q dx'ds\right)^{\frac{2}{Q}} \le C\int_{\mathbb{R}^N_+} e^{2As}|\nabla\varphi|^2 dx'ds, \tag{3.6}$$

for $B = -\frac{K(\alpha+1)}{Q}$ and $A - 1 = B - \frac{Q-2}{2Q}N = -\frac{K(\alpha+1)}{Q} - \frac{Q-2}{2Q}N$. To deduce (3.5) from (3.6) we need $2A \le -K(\alpha+1)$ which is equivalent to $2 \le \frac{Q-2}{Q}(N - K(\alpha+1))$. This last inequality is always satisfied by taking K large enough. On the other hand $BQ + 2(A-1) = -K(\alpha+1)\frac{Q+2}{Q} - \frac{Q-2}{Q}N \ne 0$, for K large.

The case $N = 2$ is treated similarly.

■

It remains to give the proof of the auxiliary results we used above. We first have

Theorem 3.2 *Let either*

$$N = 2, \qquad 2 \le Q, \qquad \text{and} \qquad B = A + 1 - \frac{2}{Q},$$

or else,

$$N \ge 3, \qquad 2 \le Q \le \frac{2N}{N-2}, \qquad \text{and} \qquad B = A + \frac{Q-2}{2Q}N.$$

Then, if $BQ + 2A \ne 0$, there exists a positive constant $C = C(A, Q, N)$ such that for any function $f \in C_0^\infty(\mathbb{R}^N_+)$ there holds

$$\left(\int_{\mathbb{R}^N_+} e^{-BQx_N}|f|^Q dx'dx_N\right)^{\frac{2}{Q}} \le C\int_{\mathbb{R}^N_+} e^{-2Ax_N}|\nabla f|^2 dx'dx_N. \tag{3.7}$$

Proof: We apply the Gagliardo–Nirenberg–Sobolev inequality (2.12) to the function $u := e^{-ax_N}v$, for any $v \in C_0^\infty(\mathbb{R}^N_+)$ and $a \ne 0$, to get

$$S_N||e^{-ax_N}v||_{L^{\frac{N}{N-1}}} \le \int_{\mathbb{R}^N_+}\left(|\nabla v|e^{-ax_N} + |a|e^{-ax_N}|v|\right)dx'dx_N\ .$$

To estimate the last term of the right hand side, we integrate by parts,

$$a\int_{\mathbb{R}^N_+} e^{-ax_N}|v|dx'dx_N = -\int_{\mathbb{R}^N_+}\nabla e^{-ax_N}|v|dx'dx_N = \int_{\mathbb{R}^N_+} e^{-ax_N}\nabla|v|dx'dx_N$$

whence,

$$|a|\int_{\mathbb{R}^N_+} e^{-ax_N}|v|dx'dx_N \le \int_{\mathbb{R}^N_+} e^{-ax_N}|\nabla v|dx'dx_N. \tag{3.8}$$

Consequently,

$$||e^{-ax_N}v||_{L^{\frac{N}{N-1}}} \leq C \int_{\mathbb{R}^N_+} e^{-ax_N}|\nabla v|dx'dx_N; \tag{3.9}$$

We note that this is true even if $a = 0$.

Using the interpolation inequality (2.16) with $w := e^{-x_N}$, as well as (3.8) and (3.9) we arrive at the following $L^p - L^1$ estimate ($e^{-(a-1)x_N} \geq e^{-ax_N}$)

$$\left(\int_{\mathbb{R}^N_+} e^{-bpx_N}|v|^p dx'dx_N\right)^{\frac{1}{p}} \leq C \int_{\mathbb{R}^N_+} e^{-(a-1)x_N}|\nabla v|dx'dx_N \ , \tag{3.10}$$

with $1 \leq p \leq \frac{N}{N-1}$, $b = a - 1 + \frac{p-1}{p}N$ and $a \neq 1$. Indeed in order to reach inequality (3.10) we need the following inequality

$$\int_{\mathbb{R}^N_+} e^{-(a-1)x_N}|v|dx'dx_N \leq C \int_{\mathbb{R}^N_+} e^{-(a-1)x_N}|\nabla v|dx'dx_N.$$

which follows from inequality (3.8) if $a \neq 1$.

We next apply (3.10) to $v := |f|^s$, $s > 0$, to obtain

$$\left(\int_{\mathbb{R}^N_+} e^{-bpx_N}|f|^{ps}dx'dx_N\right)^{\frac{1}{p}} \leq C \int_{\mathbb{R}^N_+} f^{s-1}|\nabla f|e^{-(a-1)x_N}dx'dx_N =$$

$$= C \int_{\mathbb{R}^N_+} e^{-\frac{bpx_N}{2}} f^{s-1}|\nabla f|e^{-(a-1)x_N+\frac{bpx_N}{2}}dx'dx_N \leq$$

$$\leq C \left(\int_{\mathbb{R}^N_+} e^{-bpx_N}f^{2s-2}dx'dx_N\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N_+} |\nabla f|^2 e^{-2(a-1)x_N+bpx_N}dx'dx_N\right)^{\frac{1}{2}}.$$

Choosing $s = \frac{2}{2-p}$, so that $2s - 2 = ps$ we get

$$\left(\int_{\mathbb{R}^N_+} e^{-bpx_N}|f|^{ps}dx'dx_N\right)^{\frac{2}{p}-1} \leq C \int_{\mathbb{R}^N_+} e^{(-2(a-1)+bp)x_N}|\nabla f|^2 dx'dx_N. \tag{3.11}$$

To conclude the proof of the Lemma we take $BQ = bp$, $Q = ps$, and $A = a - 1 - \frac{bp}{2}$. The condition $a \neq 1$ is equivalent to $BQ + 2A \neq 0$.

∎

As a consequence of the previous Theorem, we have the following result which is the analogue of Proposition 2.1. That is, is some cases we can remove the zero boundary condition at $x_N = 0$.

Proposition 3.3 *Let either*

$$N = 2, \qquad 2 \leq Q, \qquad \text{and} \qquad B = A + 1 - \frac{2}{Q},$$

or else,

$$N \geq 3, \qquad 2 \leq Q \leq \frac{2N}{N-2}, \qquad \text{and} \qquad B = A + \frac{Q-2}{2Q}N.$$

Then, if $BQ + 2A \neq 0$, there exists a positive constant $C = C(A, Q, N)$ such that for any function $f \in C_0^\infty(\mathcal{H}_1)$ there holds

$$\left(\int_{\mathbb{R}^N_+} e^{-BQx_N}|f|^Q dx'dx_N\right)^{\frac{2}{Q}} \leq C \int_{\mathbb{R}^N_+} e^{-2Ax_N}|\nabla f|^2 dx'dx_N. \tag{3.12}$$

Proof. To deduce (3.12) from (3.7) we will work as in the proof of Proposition 2.1 in order to remove the zero boundary condition on the hyperplane $x_N = 0$. Let $\xi(x_N)$ be a C^1 function such that $\xi(x_N) = 1$ if $x_N \geq 2$ and $\xi(x_N) = 0$ if $x_N \in [0,1]$, then for any $f \in C_0^\infty(\mathcal{H}_1)$ we have

$$\begin{aligned} LHS &:= C\left(\int_{\mathbb{R}^N_+} e^{-BQx_N}|f|^Q dx' dx_N\right)^{\frac{2}{Q}} \\ &\leq \left(\int_{\mathbb{R}^N_+} e^{-BQx_N}|f\xi|^Q dx' dx_N\right)^{\frac{2}{Q}} + \left(\int_{\mathbb{R}^N_+} e^{-BQx_N}|f(1-\xi)|^Q dx' dx_N\right)^{\frac{2}{Q}} \\ &=: I_1 + I_2. \end{aligned} \tag{3.13}$$

Applying (3.7) to the function $f\xi$, we obtain

$$I_1 \leq C\int_{\mathbb{R}^N_+} e^{-2Ax_N}\left(|\nabla f|^2 + f^2\right)\, dx' dx_N.$$

On the other hand, since the weights e^{-BQx_N} and e^{-2Ax_N} are uniformly bounded both from above and below in the interval $[0,2]$, we may apply the standard Sobolev inequality to the function $f(1-\xi)$ which is zero when $|x'| = 1$ as well as when $x_N = 2$ to get

$$I_2 \leq C\int_{\mathbb{R}^{N-1}\times[0,2]} |\nabla(f(1-\xi))|^2\, dx' dx_N \leq C\int_{\mathbb{R}^N_+} e^{-2Ax_N}\left(|\nabla f|^2 + f^2\right)\, dx' dx_N.$$

Combining the above estimates we have

$$LHS \leq C\int_{\mathbb{R}^N_+} e^{-2Ax_N}\left(|\nabla f|^2 + f^2\right)\, dx' dx_N. \tag{3.14}$$

To conclude we use the Poincaré inequality on the set $B_1' = \{x' \in \mathbb{R}^{N-1} : |x'| < 1\}$. For any fixed x_N

$$\int_{B_1'} f^2(x', x_N) dx' \leq C\int_{B_1'} |\nabla_{x'} f|^2 dx',$$

whence

$$\begin{aligned} \int_{\mathbb{R}^N_+} e^{-2Ax_N} f^2 dx' dx_N = \int_0^\infty e^{-2Ax_N}\int_{B_1'} f^2 dx' dx_N &\leq C\int_0^\infty e^{-2Ax_N}\int_{B_1'} |\nabla_{x'} f|^2 dx' dx_N\ , \\ &\leq C\int_{\mathbb{R}^N_+} e^{-2Ax_N}|\nabla f|^2 dx' dx_N. \end{aligned}$$

From this and (3.14) the result follows.

∎

We next present a new Sobolev inequality which also involves exponential weights. We used this estimate in the proof of Theorem 1.2.

Theorem 3.4 *Let either*

$$N = 2, \qquad 2 \leq Q, \qquad \text{and} \qquad B = A - \frac{2}{Q},$$

or else,

$$N \geq 3, \qquad 2 \leq Q \leq \frac{2N}{N-2}, \qquad \text{and} \qquad B = A - 1 + \frac{Q-2}{2Q} N.$$

Then, if $BQ + 2A \neq 2$, there exists a positive constant $C = C(A, Q, N)$ such that for any function $f \in C_0^\infty(\mathbb{R}^N_+)$ there holds

$$\left(\int_{\mathbb{R}^N_+} e^{BQx_N} |f|^Q dx' dx_N \right)^{\frac{2}{Q}} \leq C \int_{\mathbb{R}^N_+} e^{2Ax_N} |\nabla f|^2 dx' dx_N. \tag{3.15}$$

Proof. Working as in the proof of Theorem 3.2 we obtain (3.8) and (3.9) that is,

$$|a| \int_{\mathbb{R}^N_+} e^{ax_N} |v| dx' dx_N \leq \int_{\mathbb{R}^N_+} e^{ax_N} |\nabla v| dx' dx_N, \tag{3.16}$$

and

$$||e^{ax_N} v||_{L^{\frac{N}{N-1}}} \leq C \int_{\mathbb{R}^N_+} e^{ax_N} |\nabla v| dx' dx_N; \tag{3.17}$$

which are valid for any a in $\mathbb{R}$.

We next use the interpolation inequality (2.16) with $w := e^{x_N}$, as well as (3.16) and (3.17) to arrive at the following $L^p - L^1$ estimate ($e^{ax_N} \geq e^{(a-1)x_N}$)

$$\left(\int_{\mathbb{R}^N_+} e^{bpx_N} |v|^p dx' dx_N \right)^{\frac{1}{p}} \leq C \int_{\mathbb{R}^N_+} e^{ax_N} |\nabla v| dx' dx_N \ , \tag{3.18}$$

with $1 \leq p \leq \frac{N}{N-1}$, $b = a - 1 + \frac{p-1}{p} N$ and $a \neq 1$. To reach inequality (3.18) we used the following estimate

$$\int_{\mathbb{R}^N_+} e^{(a-1)x_N} |v| dx' dx_N \leq C \int_{\mathbb{R}^N_+} e^{(a-1)x_N} |\nabla v| dx' dx_N,$$

which is a consequence of (3.16) if $a \neq 1$.

We next apply (3.18) to $v := |f|^s$, $s > 0$, to obtain

$$\left(\int_{\mathbb{R}^N_+} e^{bpx_N} |f|^{ps} dx' dx_N \right)^{\frac{1}{p}} \leq C \int_{\mathbb{R}^N_+} f^{s-1} |\nabla f| e^{ax_N} dx' dx_N =$$

$$= C \int_{\mathbb{R}^N_+} e^{\frac{bpx_N}{2}} f^{s-1} |\nabla f| e^{ax_N - \frac{bpx_N}{2}} dx' dx_N \leq$$

$$\leq C \left(\int_{\mathbb{R}^N_+} e^{bpx_N} f^{2s-2} dx' dx_N \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N_+} |\nabla f|^2 e^{2ax_N - bpx_N} dx' dx_N \right)^{\frac{1}{2}}.$$

Choosing $s = \frac{2}{2-p}$, so that $2s - 2 = ps$ we get

$$\left(\int_{\mathbb{R}^N_+} e^{bpx_N} |f|^{ps} dx' dx_N \right)^{\frac{2}{p}-1} \leq C \int_{\mathbb{R}^N_+} e^{(2a-bp)x_N} |\nabla f|^2 dx' dx_N. \tag{3.19}$$

To conclude the proof of the Lemma we take $BQ = bp$, $Q = ps$, and $A = a - \frac{bp}{2}$. The condition $a \neq 1$ is equivalent to $BQ + 2A \neq 2$.

∎

We finally have

Proposition 3.5 *Let either*

$$N = 2, \qquad 2 \le Q, \qquad \text{and} \qquad B = A - \frac{2}{Q},$$

or else,

$$N \ge 3, \qquad 2 \le Q \le \frac{2N}{N-2}, \qquad \text{and} \qquad B = A - 1 + \frac{Q-2}{2Q}N.$$

Then, if $BQ + 2A \neq 2$, there exists a positive constant $C = C(A, Q, N)$ such that for any function $f \in C_0^\infty(\mathcal{H}_1)$ there holds

$$\left(\int_{\mathbb{R}_+^N} e^{BQx_N} |f|^Q dx' dx_N\right)^{\frac{2}{Q}} \le C \int_{\mathbb{R}_+^N} e^{2Ax_N} |\nabla f|^2 dx' dx_N. \tag{3.20}$$

Proof: We need to remove the zero boundary condition of f, on the hyperplane $x_N = 0$. As usual, let $\xi(x_N)$ be a C^1 function such that $\xi(x_N) = 1$ if $x_N \ge 2$ and $\xi(x_N) = 0$ if $x_N \in [0, 1]$, then for any $f \in C_0^\infty(\mathcal{H}_1)$ we have $f = f\xi + f(1 - \xi)$. To conclude the proof we argue as in the proof of Proposition 3.3. We omit further details.

4 The supercritical case $\sigma > 2$

In this Section we will give the proof of Theorem 1.3. It is a direct consequence of a more general result. We recall that

$$Q_{cr} = Q_{cr}(N, \alpha, \sigma) := \frac{2\left(N + \frac{2\alpha+\sigma}{2-\sigma}\right)}{N + \frac{2\alpha+\sigma}{2-\sigma} - 2}.$$

We also set

$$\bar{Q}(N, \alpha, \sigma) := \frac{2\left(N - \frac{2\alpha+\sigma}{2-\sigma}\right)}{N + \frac{2\alpha+\sigma}{2-\sigma} - 2},$$

and

$$\theta_{cr} := \frac{2-\sigma}{2}\left[\frac{Q}{2}\left\{N + \frac{2\alpha+\sigma}{2-\sigma} - 2\right\} - \left\{N + \frac{2\alpha+\sigma}{2-\sigma}\right\}\right] = \frac{2-\sigma}{4}\left(N + \frac{2\alpha+\sigma}{2-\sigma} - 2\right)(Q - Q_{cr}). \tag{4.1}$$

We then have

Theorem 4.1 *Let $N \ge 2$, $\alpha < -1$ and $\sigma \in (2, -2\alpha)$. Then, for any $\theta \ge \theta_{cr}$ and any $Q \neq \bar{Q}$ with $2 \le Q \le \frac{2N}{N-2}$, in case $N \ge 3$, or $Q \ge 2$ in case $N = 2$, there exists a positive constant $C = C(Q, N, \alpha, \sigma, \theta)$, such that for any function $f \in C_0^\infty(\mathcal{C}_1)$ there holds*

$$\left(\int_{\mathcal{C}_1} (1 - |\lambda|)^{\alpha+\theta} |f(x', \lambda)|^Q dx' d\lambda\right)^{\frac{2}{Q}} \le C \int_{\mathcal{C}_1} (1 - |\lambda|)^\alpha \left(|\nabla_{x'} f|^2 + (1 - |\lambda|)^\sigma |\partial_\lambda f|^2\right) dx' d\lambda\ . \tag{4.2}$$

To prove the above result we will use the following consequence of Theorem 1.4

Proposition 4.2 *Let $\mathcal{H}_1 = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x'| < 1\}$,*

$$N \ge 3, \qquad 2 \le Q \le \frac{2N}{N-2}, \qquad \text{and} \qquad B = A - 1 + \frac{Q-2}{2Q}N,$$

or

$$N = 2, \qquad 2 \le Q, \qquad \text{and} \qquad B = A - \frac{2}{Q}.$$

If $BQ + 2A \neq 0$, or if $A = B = 0$ then, there exists a positive constant $C = C(A, Q, N)$, such that for any function $f \in C_0^\infty(\mathcal{H}_1)$ there holds

$$\left(\int_{\{x_N > 1\}} x_N^{BQ} |f(x', x_N)|^Q dx' dx_N \right)^{\frac{2}{Q}} \le C \int_{\{x_N > 1\}} x_N^{2A} \left(|\nabla_{x'} f|^2 + |\partial_{x_N} f|^2 \right) \, dx' dx_N \, . \tag{4.3}$$

Proof of Proposition 4.2: The proof is quite similar to the proof of Proposition 2.1 we therefore sketch it. We use a C^1 cutoff function $\xi(x_N)$ such that $\xi(x_N) = 1$ in $x_N \ge 2$ and $\xi(x_N) = 0$ if $0 \le x_N \le 1$. Hence we write $f = f\xi + f(1-\xi)$. Now $f(1-\xi)$ satisfies the standard Sobolev inequality in $1 \le x_N \le 2$, while $f\xi$ satisfies the assumptions of Theorem 1.4 part (i). Putting things together and using Poincaré inequality in the x'–variables we conclude the proof. We omit further details.

∎

Proof of Theorems 4.1 and 1.3: We first prove Theorem 4.1. As usual, it is enough to prove (4.2) in the upper half cylinder. That is, if $f \in C_0^\infty(\mathcal{C}_1)$ then we need to show that

$$\left(\int_{\{0<\lambda<1\}} (1-\lambda)^{\alpha+\theta} |f(x', \lambda)|^Q dx' d\lambda \right)^{\frac{2}{Q}} \le C \int_{\{0<\lambda<1\}} (1-\lambda)^{\alpha} \left(|\nabla_{x'} f|^2 + (1-\lambda)^{\sigma} |\partial_\lambda f|^2 \right) dx' d\lambda \, . \tag{4.4}$$

As in the proof of Theorem 1.1, we change variables by $x' = x'$, $s = (1-\lambda)^{\frac{2-\sigma}{2}}$ thus setting $\varphi(x', s) := f(x', 1 - s^{\frac{2}{2-\sigma}})$, it follows that inequality (4.4) is equivalent to

$$\left(\int_{\{s>1\}} s^{\frac{\sigma+2\alpha+2\theta}{2-\sigma}} |\varphi(x', s)|^Q dx' ds \right)^{\frac{2}{Q}} \le C \int_{\{s>1\}} s^{\frac{\sigma+2\alpha}{2-\sigma}} \left(|\nabla_{x'} \varphi|^2 + |\partial_s \varphi|^2 \right) \, dx' ds \, , \tag{4.5}$$

for $\varphi \in C_0^\infty(\mathcal{H}_1)$. We now use Proposition 4.2. Suppose first that $N \ge 3$. For $A = \frac{\sigma+2\alpha}{2(2-\sigma)}$ and $B = \frac{\sigma+2\alpha}{2(2-\sigma)} - 1 + \frac{Q-2}{2Q} N$, with $2 \le Q \le \frac{2N}{N-2}$ we have that the right hand side of (4.5) dominates

$$\left(\int_{\{s>1\}} s^{BQ} |\varphi(x', s)|^Q dx' ds \right)^{\frac{2}{Q}} .$$

To deduce (4.5) we need $\frac{\sigma+2\alpha+2\theta}{2-\sigma} \le BQ = \left(\frac{\sigma+2\alpha}{2(2-\sigma)} - 1 + \frac{Q-2}{2Q} N \right) Q$, which is satisfied by any $\theta \ge \theta_{cr}$ as defined in (4.1).

Let us finally observe that $BQ + 2A \neq 0$ corresponds to the assumption $\theta \neq -\sigma - 2\alpha$ that is $Q \neq \bar{Q}$.

The case $N = 2$ is treated quite similarly.

To prove Theorem 1.3 we note that for $Q \ge Q_{cr}(N, \alpha, \sigma)$ we have that $\theta_{cr} \le 0$ and therefore we can take $\theta = 0$.

∎

5 The case of codimension k degeneracy $1 < k < N$.

In this section we will prove Theorems 1.6 and 1.7.
Proof of Theorem 1.7: We will divide the proof into three steps.

step 1 (The critical L^1 weighted anisotropic inequality). Suppose that either $\beta > 0$ and $u \in C_0^\infty(\mathbb{R}^N)$ or else $\beta \in \mathbb{R}$ and $u \in C_0^\infty(\mathbb{R}^N_+)$. Then, for a constant C depending only on N there holds:

$$\left(\int_{\mathbb{R}^N_+} x_N^{\frac{\beta N+\gamma(k-1)}{N-1}} |u|^{\frac{N}{N-1}} dx\right)^{\frac{N-1}{N}} \leq C(N) \int_{\mathbb{R}^N_+} x_N^\beta \left(|\nabla_{x',x_N} u| + x_N^\gamma |\nabla_y u|\right) dx. \tag{5.1}$$

The proof follows closely the standard proof of the L^1 Gagliardo– Nirenberg–Sobolev inequality. Suppose that $\beta \neq 0$ and $u \in C_0^\infty(\mathbb{R}^N_+)$. Let us write $x' = (x'_1, \ldots, x'_{N-k})$ and $y = (y_1, \ldots, y_{k-1})$. We then have that for $i = 1, \ldots, N-k$,

$$u(x) = -\int_{x'_i}^\infty u_{x'_i}(x'_1, ...t_i, ..., x'_{N-k}, x_N, y) dt_i,$$

From which it follows easily that

$$x_N^\beta |u(x)| \leq \int_{\mathbb{R}} x_N^\beta |u_{x'_i}| dt_i. \tag{5.2}$$

We similarly have that

$$x_N^{\gamma+\beta} |u(x)| \leq \int_{\mathbb{R}} x_N^{\gamma+\beta} |u_{y_i}| ds_i, \tag{5.3}$$

where integration is performed in the y_i–variable, $i = 1, \ldots, k-1$. A similar argument shows that

$$x_N^\beta |u(x)| \leq \int_0^\infty (\xi^\beta |u_{x_N}| + |\beta| \xi^{\beta-1} |u|) d\xi,$$

from which it follows easily that

$$x_N^\beta |u(x)| \leq 2 \int_0^\infty \xi^\beta |u_{x_N}| d\xi \tag{5.4}$$

which is true also if $\beta = 0$.

Multiplying (5.2), (5.3), (5.4) and raising to the power $\frac{1}{N-1}$ we get

$$x_N^{\frac{\beta N+\gamma(k-1)}{N-1}} |u(x)|^{\frac{N}{N-1}} \leq 2 \left(\prod_{i=1}^{N-k} \int_{\mathbb{R}} x_N^\beta |u_{x'_i}| dt_i \left(\int_0^\infty \xi^\beta |u_{x_N}| d\xi\right) \prod_{j=1}^{k-1} \int_{\mathbb{R}} x_N^{\gamma+\beta} |u_{y_j}| ds_j\right)^{\frac{1}{N-1}}. \tag{5.5}$$

We next integrate with respect to x'_1 and apply Holder's inequality in the right hand side, then we integrate with respect to the x'_2 variable and so on until we integrate with respect to all variables. This way we reach the following estimate

$$\int_{\mathbb{R}^N_+} x_N^{\frac{\beta N+\gamma(k-1)}{N-1}} |u(x)|^{\frac{N}{N-1}} dx \leq 2 \left(\prod_{i=1}^{N-k} \int_{\mathbb{R}^N_+} x_N^\beta |u_{x'_i}| dx \left(\int_{\mathbb{R}^N_+} x_N^\beta |u_{x_N}| dx\right) \prod_{j=1}^{k-1} \int_{\mathbb{R}^N_+} x_N^{\gamma+\beta} |u_{y_j}| dx\right)^{\frac{1}{N-1}}. \tag{5.6}$$

To continue we use in the right hand side of (5.6) the well known inequality

$$\prod_{i=1}^N a_i \leq \frac{1}{N^N} \left(\sum_{i=1}^N a_i\right)^N, \qquad a_i \geq 0.$$

We then conclude that

$$\int_{\mathbb{R}^N_+} x_N^{\frac{\beta N+\gamma(k-1)}{N-1}} |u(x)|^{\frac{N}{N-1}} dx \leq C(N) \left(\int_{\mathbb{R}^N_+} x_N^\beta \left(|\nabla_{x',x_N} u| + x_N^\gamma |\nabla_y u|\right) dx\right)^{\frac{N}{N-1}}, \tag{5.7}$$

which is the sought for estimate (5.1).

step 2 (The L^p–L^1 estimate). For $1 \le p \le \frac{N}{N-1}$ we will use the interpolation inequality (2.16) with weight

$$w = x_N^{\frac{N+\gamma(k-1)}{N}},$$

and

$$a := \frac{\beta N + \gamma(k-1)}{N + \gamma(k-1)}, \qquad\qquad b = a - 1 + \frac{p-1}{p}N.$$

For these choices we have that

$$\|w^b u\|_{L^p} \le C_1 \left(\int_{\mathbb{R}^N_+} x_N^{\frac{\beta N + \gamma(k-1)}{N-1}} |u(x)|^{\frac{N}{N-1}} dx \right)^{\frac{N-1}{N}} + C_2 \int_{\mathbb{R}^N_+} x_N^{\beta-1} |u| dx. \tag{5.8}$$

We will also make use of the estimate

$$|\beta| \int_{\mathbb{R}^N_+} x_N^{\beta-1} |u| dx \le \int_{\mathbb{R}^N_+} x_N^{\beta} |u_{x_N}| dx \le \int_{\mathbb{R}^N_+} x_N^{\beta} |\nabla_{x',x_N} u| dx \tag{5.9}$$

which follows easily using an integration by parts if $\beta \neq 0$. From (5.7), (5.8), (5.9) and using the specific values of the weight and the parameters we get

$$\|x_N^{\tilde{b}} u\|_{L^p} \le C \int_{\mathbb{R}^N_+} x_N^{\beta} \left(|\nabla_{x',x_N} u| + x_N^{\gamma} |\nabla_y u| \right) dx, \tag{5.10}$$

with $\gamma \in \mathbb{R}$, $\beta \neq 0$ and

$$\tilde{b} = \beta - 1 + \frac{p-1}{p}(N + \gamma(k-1)). \tag{5.11}$$

step 3 (The L^Q–L^2 estimate). Here we will apply estimate (5.10) to the function $u(x) = |f(x)|^s$ with $s > 0$. After some elementary calculations and use of Holder's inequality we find that

$$\left(\int_{\mathbb{R}^N_+} x_N^{\tilde{b}p} |f|^{sp} dx \right)^{\frac{1}{p}} \le C \left(\int_{\mathbb{R}^N_+} x_N^{\tilde{b}p} |f|^{2s-2} dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N_+} x_N^{2\beta - \tilde{b}p} \left(|\nabla_{x',x_N} f|^2 + x_N^{2\gamma} |\nabla_y f|^2 \right) dx \right)^{\frac{1}{2}}. \tag{5.12}$$

We now choose $s = \frac{2}{2-p}$ (so that $sp = 2s-2$), $Q = sp$, $BQ = \tilde{b}p$ and $2A = 2\beta - \tilde{b}p$. For this choices we get that

$$\left(\int_{\mathbb{R}^N_+} x_N^{BQ} |f(x)|^Q dx \right)^{\frac{2}{Q}} \le C \int_{\mathbb{R}^N_+} x_N^{2A} \left(|\nabla_{x',x_N} f|^2 + x_N^{2\gamma} |\nabla_y f|^2 \right) \, dx \; .$$

with $\gamma \in \mathbb{R}$, $2 \le Q \le \frac{2N}{N-2}$ if $N \ge 3$ or for $Q \ge 2$ if $N = 2$, and $B = A - 1 + \frac{Q-2}{2Q}(N + \gamma(k-1))$. The condition $\beta \neq 0$ is equivalent to $2A + BQ \neq 0$.

The case where $f \in C_0^\infty(\mathbb{R}^N)$ and $2A + BQ > 0$, or equivalently $\beta > 0$ is practically the same; we just note that (5.8) remains true for $\beta > 0$.

■

We are now ready to give the proof of Theorem 1.6.

Proof of Theorem 1.6: We will use a (finite) partition of unity for Ω which we denote by φ_i, $i = 0, \cdots, m$, such that $1 = \sum_{i=0}^m \varphi_i^2$. We denote by Ω_i the support of each function φ_i. We assume $\Omega_0 \subset\subset \Omega$ and therefore $c \le d(\lambda, \partial\Omega) \le c^{-1}$ for $\lambda \in \Omega_0$. For $i \ge 1$, in each Ω_i we will use local coordinates (y^i, x_N^i),

$i \in \{1, \cdots, m\}$ with $y^i \in \Delta_i := \{y^i : |y^i_j| \leq \beta$ for $j = 1, \cdots, k-1\}$ for some positive constant $\beta < 1$. Each point $\lambda \in \bar{\Omega}_i \cap \partial\Omega$ is described by $\lambda = (y^i, a_i(y^i))$, where the functions a_i satisfy a Lipschitz condition on $\overline{\Delta}_i$ with a constant $A > 0$ that is

$$|a_i(y^i) - a_i(z^i)| \leq A|y^i - z^i|$$

for $y^i, z^i \in \overline{\Delta}_i$; We next define $\hat{B}_i$ by $\hat{B}_i := \{(y^i, x^i_N) : y^i \in \Delta_i, a_i(y^i) - \beta < x^i_N < a_i(y^i) + \beta\}$ so that $\hat{B}_i \cap \Omega = \{(y^i, x^i_N) : y^i \in \Delta_i, a_i(y^i) - \beta < x^i_N < a_i(y^i)\}$ and $\Gamma_i = \hat{B}_i \cap \partial\Omega = \{(y^i, x^i_N) : y^i \in \Delta_i, x^i_N = a_i(y^i)\}$. We note that $\Omega_i \subset \hat{B}_i \cap \Omega$. Next we observe that for any $y \in \hat{B}_i \cap \Omega$ we have that $(1+A)^{-1}(a_i(y^i) - x^i_N) \leq d(\lambda) \leq (a_i(y^i) - x^i_N)$, (see, e.g., Corollary 4.8 in [K]). By straightening the boundary Γ_i we may suppose that $\Gamma_i \subset \{x^i_N = 0\}$. From now on we omit the subscript i for convenience.

As a first step we will prove that for $u \in C_0^\infty(B_1 \times H_1^+)$, where $B_1 := \{|x'| < 1\}$ and $H_1^+ := \{|y'| < 1\} \times \{0 < x_N < 1\}$ there holds

$$\left(\int_{B_1 \times H_1^+} x_N^\alpha |u|^Q dx' dx_N dy\right)^{\frac{2}{Q}} \leq C \int_{B_1 \times H_1^+} x_N^\alpha \left(|\nabla_{x'} u|^2 + x_N^\sigma |\nabla_y u|^2 + x_N^\sigma |\partial_{x_N} u|^2\right) dx' dx_N dy, \tag{5.13}$$

where $x' \in \mathbb{R}^{N-k}$, and $\lambda = (y, x_N)$ with $y \in \mathbb{R}^{k-1}$ and $x_N \in \mathbb{R}$. We change variables by $t = x_N^{\frac{2-\sigma}{2}}$ thus obtaining

$$\left(\int_{\{0<t<1\}} t^{\frac{2\alpha+\sigma}{2-\sigma}} |u|^Q dx' dt dy\right)^{\frac{2}{Q}} \leq C \int_{\{0<t<1\}} t^{\frac{2\alpha+\sigma}{2-\sigma}} \left(|\nabla_{x',t}\, u|^2 + t^{\frac{2\sigma}{2-\sigma}} |\nabla_y u|^2\right) dx' dt dy\ . \tag{5.14}$$

Estimate (5.14) follows from Theorem 1.7 part (i) taking $2A = BQ = \frac{2\alpha+\sigma}{2-\sigma}$ and $2\gamma = \frac{2\sigma}{2-\sigma}$. We note that $2A + BQ \neq 0$ is equivalent to $2\alpha + \sigma \neq 0$. Also, $Q^k_{cr} \leq \frac{2N}{N-2}$ since $2\alpha + k\sigma \geq 0$.

Next, for $f(x) = \sum_{i=0}^m \varphi_i(\lambda) f(x)$, we write

$$\left(\int_{B_1 \times \Omega} d^\alpha |f|^Q dx\right)^{\frac{2}{Q}} \leq C \sum_{i=0}^m \left(\int_{B_1 \times \Omega} d^\alpha |f\varphi_i|^Q dx\right)^{\frac{2}{Q}} \tag{5.15}$$

Using (5.13) for $i = 1, \ldots, m$ and the standard Sobolev inequality for $i = 0$, in the right hand side of (5.15), after some calculations, we end up with

$$\left(\int_{B_1 \times \Omega} d^\alpha |f|^Q dx\right)^{\frac{2}{Q}} \leq C \int_{B_1 \times \Omega} d^\alpha \left(|\nabla_{x'} f|^2 + d^\sigma |\nabla_\lambda f|^2\right) dx + \int_{B_1 \times \Omega} d^{\alpha+\sigma} f^2 dx. \tag{5.16}$$

To estimate the last term, we first use Proposition 5.1 to obtain

$$\int_{B_1 \times \Omega} d^{\alpha+\sigma} f^2 dx \leq C \left(\int_{B_1 \times \Omega} d^{\alpha+\sigma} |\nabla_\lambda f|^2 dx + \int_{B_1 \times \Omega} d^\alpha f^2 dx\right), \tag{5.17}$$

and then Poincare inequality in the x'–variables,

$$\int_{B_1 \times \Omega} d^\alpha f^2 dx \leq \int_{B_1 \times \Omega} d^\alpha |\nabla_{x'} f|^2 dx.$$

Hence, we end up with

$$\int_{B_1 \times \Omega} d^{\alpha+\sigma} f^2 dx \leq C \int_{B_1 \times \Omega} d^\alpha \left(|\nabla_{x'} f|^2 + d^\sigma |\nabla_\lambda f|^2\right) dx.$$

Combining this with (5.16) we conclude the result.

■

We next prove the Proposition we used in the proof of Theorem 1.6.

Proposition 5.1 *Let $\alpha+\sigma<1$. Then there exists a constant $C=C(\alpha,\sigma,\Omega)>0$ such that*

$$\int_\Omega d^{\alpha+\sigma-2}f^2 d\lambda \le C\int_\Omega d^{\alpha+\sigma}|\nabla_\lambda f|^2 d\lambda \quad , \quad \forall\ f\in C_0^\infty(\Omega); \tag{5.18}$$

The previous inequality fails when $\alpha+\sigma\ge 1$.

Let $\alpha+\sigma=1$. Then there exists a constant $C=C(\alpha,\Omega)>0$ such that

$$\int_\Omega \frac{X^2(d)}{d}f^2 d\lambda \le C\left[\int_\Omega d|\nabla_\lambda f|^2 d\lambda+\int_\Omega d^\alpha f^2 d\lambda\right] \quad , \quad \forall\ f\in C_0^\infty(\Omega), \tag{5.19}$$

where $X(d):=(1-\ln(d/D))^{-1}$ and $D:=\sup_{\lambda\in\Omega}d(\lambda)$.

Finally, if $\alpha+\sigma>1$, there exists a constant $C=C(\alpha,\sigma,\Omega)>0$ such that

$$\int_\Omega d^{\alpha+\sigma-2}f^2 d\lambda \le C\left[\int_\Omega d^{\alpha+\sigma}|\nabla_\lambda f|^2 d\lambda+\int_\Omega d^\alpha f^2 d\lambda\right] \quad , \quad \forall\ f\in C_0^\infty(\Omega). \tag{5.20}$$

Proof of Proposition 5.1:

step 1: An auxiliary estimate: Let $\Omega_\delta:=\{\lambda\in\Omega: d(\lambda)\le\delta\}$. We will establish the following estimate: Given any $\epsilon>0$ there exists a $\delta_0>0$ such that for any $0<\delta\le\delta_0$ and any $u\in C_0^\infty(\Omega)$

$$\int_{\Omega_\delta}|\nabla_\lambda u|^2 d\lambda \ge \frac{1}{4}\int_{\Omega_\delta}\frac{u^2}{d^2}d\lambda+\left(\frac{1}{4}-\epsilon\right)\int_{\Omega_\delta}\frac{X^2(d)u^2}{d^2}d\lambda+\frac{1-X(\delta)}{2\delta}\int_{\partial\Omega_\delta}u^2 dS. \tag{5.21}$$

To prove this our starting point is the obvious relation

$$0\le\int_{\Omega_\delta}\left|\nabla_\lambda u-\left(\frac{\nabla d}{2d}-\frac{X\nabla d}{2d}\right)u\right|^2 d\lambda.$$

Expanding the square, integrating by parts and using the fact that $|d\Delta d|$ can be made arbitrarily small in Ω_δ, for δ sufficiently small, the result follows.

step 2: Proof of (5.18) We change variables by $u:=d^{\frac{\alpha+\sigma}{2}}f$. A straightforward calculation leads to the following identity

$$\int_{\Omega_\delta}d^{\alpha+\sigma}|\nabla_\lambda f|^2 d\lambda=\int_{\Omega_\delta}|\nabla_\lambda u|^2 d\lambda-\frac{\alpha+\sigma}{2}\left(1-\frac{\alpha+\sigma}{2}\right)\int_{\Omega_\delta}\frac{u^2}{d^2}d\lambda+\frac{\alpha+\sigma}{2}\int_{\Omega_\delta}\frac{\Delta d}{d}u^2 d\lambda-\frac{\alpha+\sigma}{2\delta}\int_{\partial\Omega_\delta}u^2 dS \tag{5.22}$$

From (5.21), (5.22) and using the fact that $|\Delta d|<C$ in Ω_δ, we easily get that there exist positive constants c such that for δ sufficiently small

$$\int_{\Omega_\delta}d^{\alpha+\sigma}|\nabla_\lambda f|^2 d\lambda\ge c\int_{\Omega_\delta}d^{\alpha+\sigma-2}f^2 d\lambda+\frac{c}{\delta^{1-(\alpha+\sigma)}}\int_{\partial\Omega_\delta}f^2 dS. \tag{5.23}$$

On the other hand, away from the boundary we have that

$$\int_{\Omega\setminus\Omega_\delta}f^2\le C\int_{\Omega\setminus\Omega_\delta}|\nabla_\lambda f|^2+C\int_{\partial\Omega_\delta}f^2 dS,$$

from which one can easily deduce

$$\int_{\Omega\setminus\Omega_\delta}d^{\alpha+\sigma-2}f^2\le C_\delta\int_{\Omega\setminus\Omega_\delta}d^{\alpha+\sigma}|\nabla_\lambda f|^2+\frac{C_\delta}{\delta^{1-(\sigma+\alpha)}}\int_{\partial\Omega_\delta}f^2 dS. \tag{5.24}$$

Combining (5.23) and (5.24) the result follows.

We note that when $\alpha+\sigma \geq 1$, the constants can be approximated by $C_0^\infty(\Omega)$ functions in the norm given by $\|v\|_{H^1(d^{\alpha+\sigma})} := \int_\Omega d^{\alpha+\sigma}(|\nabla v|^2+v^2)d\lambda$; see Theorem 2.11 of [FMT]. In particular, one can put a constant function in (5.18) to obtain an obvious contradiction.

step 3: Proof of (5.19) and (5.20) We first give the proof of (5.19). For $g \in C_0^\infty(\Omega_\delta)$ and $u = d^{\frac{1}{2}}g$ we get from (5.21) and (5.22) that for any $\epsilon > 0$ there exists a $\delta_0 > 0$ such that for any $0 < \delta \leq \delta_0$,

$$\int_{\Omega_\delta} d|\nabla_\lambda g|^2 d\lambda \geq \left(\frac{1}{4} - 2\epsilon\right) \int_{\Omega_\delta} \frac{X^2(d)g^2}{d} d\lambda. \tag{5.25}$$

To establish the result we argue as follows. Let $\xi(s)$ be a C^1 function such that $0 \leq \xi \leq 1$, $\xi(s) = 0$ if $s \geq 2$, $\xi(s) = 1$ if $0 \leq s \leq 1$, and let us define $\varphi(\lambda) = \xi\left(\frac{d}{\delta}\right)$.

Whence for $f \in C_0^\infty(\Omega)$ we have that $f = \varphi f + (1-\varphi)f$. Using (5.25) for φf and the fact that on the support of $(1-\varphi)f$ we have $\frac{X^2(d)}{d} \leq \delta^{-1}$ and $d^\alpha \geq min\{\delta^\alpha, D^\alpha\}$, we arrive at

$$\begin{aligned} \int_\Omega \frac{X^2(d)f^2}{d} d\lambda &\leq c_1 \int_\Omega d(\varphi^2|\nabla_\lambda f|^2 + |\nabla_\lambda \varphi|^2 f^2)d\lambda + c_2 \int_\Omega (1-\varphi)^2 f^2 d\lambda \\ &\leq C\left(\int_\Omega d|\nabla_\lambda f|^2 d\lambda + \int_\Omega d^\alpha f^2 d\lambda\right). \end{aligned} \tag{5.26}$$

To prove (5.20) we work similarly. We just note that the analogue of (5.25) is

$$\int_{\Omega_\delta} d^{\alpha+\sigma}|\nabla_\lambda g|^2 d\lambda \geq \left(\frac{1-(\alpha+\sigma)}{2}\right)^2 \int_{\Omega_\delta} d^{\alpha+\sigma-2} g^2 d\lambda, \qquad g \in C_0^\infty(\Omega_\delta).$$

We omit further details.

∎

Remark. The limit case $k = N$ of Theorem 1.6, corresponds to the following isotropic weighted inequality:

$$\left(\int_\Omega d^\alpha |f|^Q d\lambda\right)^{\frac{2}{Q}} \leq C \int_\Omega d^{\alpha+\sigma}|\nabla_\lambda f|^2 d\lambda, \tag{5.27}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $f \in C_0^\infty(\Omega)$. Using similar arguments one can show that the above inequality is true if $\alpha+\sigma < 1$, provided that $\alpha > -1$, $2\alpha + N\sigma \geq 0$ and $2 \leq Q \leq Q_{cr}^N = \frac{2(N+\alpha)}{N+\alpha+\sigma-2}$. On the other hand inequality (5.27) fails if $\alpha+\sigma \geq 1$, by the argument of Proposition 5.1.

We finally have the following analogue of Corollary 1.5.

Corollary 5.2 *For $N \geq 3$, $1 < k < N$, $m > -1$ and $\epsilon \in (0, \frac{1}{2})$ we set*

$$C_{1,\epsilon} := \{(x', \lambda) \in \mathbb{R}^{N-k} \times \mathbb{R}^k : |x'| < 1,\ |\lambda| < 1 - \epsilon^{1+m}\}\ .$$

Let $\alpha > -1$ and $\beta > 0$ satisfy

$$-2\alpha(1+m) < \beta m < 2(1+m) \qquad \text{and} \qquad 2\alpha(1+m) + \beta k m \geq 0.$$

Then, for any P with

$$2 \leq P \leq P_{cr}(N,m,\alpha,\beta,k) := \frac{2\left(N + \frac{2\alpha(1+m)+\beta k m}{2(1+m)-\beta m}\right)}{N + \frac{2\alpha(1+m)+\beta k m}{2(1+m)-\beta m} - 2},$$

there exists a positive constant $C = C(N, P, m, \alpha, \beta, k)$ independent of ϵ, such that for any function $f \in C_0^\infty(C_{1,\epsilon})$ there holds

$$\left(\int_{C_{1,\epsilon}} (1-|\lambda|)^\alpha |f(x',\lambda)|^P dx' d\lambda\right)^{\frac{2}{P}} \leq C \int_{C_{1,\epsilon}} (1-|\lambda|)^\alpha \left(|\nabla_{x'} f|^2 + \frac{(1-|\lambda|)^\beta}{\epsilon^\beta} |\nabla_\lambda f|^2\right) dx' d\lambda.$$

Proof: It follows from Theorem 1.6. We have that $1 - |\lambda| > \epsilon^{1+m}$, that is $\epsilon^{-1} > (1-|\lambda|)^{-\frac{1}{1+m}}$ and consequently $\frac{(1-|\lambda|)^\beta}{\epsilon^\beta} > (1-|\lambda|)^{\frac{\beta m}{1+m}}$, $\beta > 0$. The result then follows from Theorem 1.6 by choosing $\sigma := \frac{\beta m}{1+m}$ there; in particular $P_{cr}(N, m, \alpha, \beta, k) = Q_{cr}(N, \alpha, \frac{\beta m}{1+m}, k)$.

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